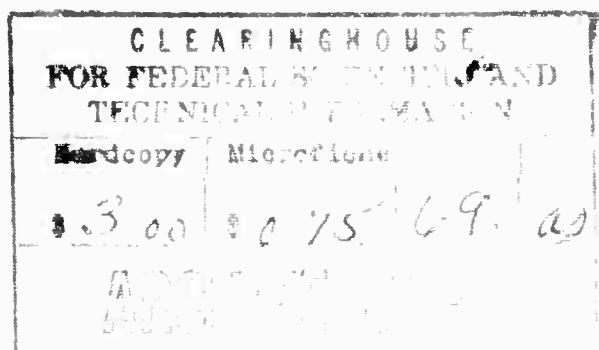


MEMORANDUM
RM-4679-ARPA
SEPTEMBER 1965

PERTURBATIONS OF A GRAVITY
GRADIENT STABILIZATION SYSTEM

R. H. Frick



PREPARED FOR:
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PREFACE

This Memorandum was prepared for the Vela Analysis project under the Advanced Research Projects Agency's contract with RAND. The project is a broad study of systems to detect above-ground nuclear bursts, and of necessity includes research into satellite systems. The Memorandum should assist in determining whether or not gravity gradient attitude stabilization is adequate for particular missions under consideration.

SUMMARY

Satellite attitude stabilization systems utilizing the gradient of the earth's gravitational field have the advantage of being completely passive (i.e., no power is required). However, the stabilizing moments developed are extremely small and as a result such systems are subject to attitude perturbations from sources which would ordinarily be neglected in the evaluation of an active stabilization system.

This Memorandum investigates certain of these sources of attitude perturbation including micro-meteoroid impact, solar radiation pressure, station-keeping propulsion, orbital eccentricity, and on-board rotating machinery. In the analysis, formulas are developed which determine the magnitude of the pitch, roll and yaw perturbations resulting from each of the above sources. Application of these results to an assumed vehicle configuration shows that the resulting attitude perturbations can be of the order of several degrees. While these perturbations do not necessarily rule out gravity gradient stabilization, their magnitudes for a given vehicle configuration should be considered for their compatibility with the mission requirements.

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LIST OF SYMBOLS

A	presented area of vehicle
a_y , b_y , c_y	direction cosines of Y axis in the x, y, z system
C_o	orbital angular momentum
E	voltage on motor
F	motor parameter
F_1 , F_2 , F_3 , F_4	functions of ω_1 and ω_2
F_R	force due to solar radiation
f	viscous damping constant
G_1 , G_2	functions of ω_1 and ω_2
G_p , G_Y , G_R	internal angular momenta
g_o	gravitational acceleration
H_1 , H_2 , H_3	functions of ω_1 and ω_2
I_p , I_Y , I_R	impulse moments
I_s	spin moment of inertia of rotor
I_T	transverse moment of inertia of rotor
I_x , I_y , I_z	principal moments of inertia of vehicle
I'_x , I'_y , I'_z	principal moments of inertia of vehicle and rotor combined
k_1	back emf constant of motor
k_o	torque constant of motor
M_x , M_y , M_z	applied torque components
M_p , M_Y , M_R	step moments
m	meteoroid mass
p_o	radiation pressure
R	motor resistance
R_o	earth radius

R_x	inertia function, $(I_z - I_y)/I_x$
R_y	inertia function, $(I_z - I_x)/I_y$
R_z	inertia function, $(I_x - I_y)/I_z$
r	distance to satellite from earth's center
r_c	orbital radius
r_p, r_Y, r_R	moment arms
T_o	motor time constant
t	time
v	meteoroid velocity
X, Y, Z	orbital coordinate system
x, y, z	body coordinate system
α	pitch angle
α_m	oscillatory amplitude of α
α_{ss}	steady-state value of α
$\dot{\alpha}_2$	motor angular velocity (pitch)
$\dot{\alpha}_{2ss}$	steady-state value of $\dot{\alpha}_2$
β	roll angle
β_m	oscillatory amplitude of β
β_{ss}	steady-state value of β
$\dot{\beta}_2$	motor angular velocity (roll)
$\dot{\beta}_{2ss}$	steady-state value of $\dot{\beta}_2$
$\delta(\circ)$	Dirac delta function
ϵ	orbital eccentricity
θ	orbital angle
$\dot{\theta}$	instantaneous orbital angular rate
μ_β	coupling constant
μ_ψ	coupling constant

ψ	yaw angle
ψ_m	oscillatory amplitude of ψ
ψ_{ss}	steady-state value of ψ
$\dot{\psi}_2$	motor angular velocity (yaw)
$\dot{\psi}_{2ss}$	steady-state value of $\dot{\psi}_2$
ω_o	orbital angular rate
$\omega_x, \omega_y, \omega_z$	vehicle angular velocity components
ω_{x2}	total angular velocity of roll rotor
ω_{y2}	total angular velocity of yaw rotor
ω_{z2}	total angular velocity of pitch rotor
ω_α	pitch natural frequency
ω_β	roll natural frequency (uncoupled)
ω_ψ	yaw natural frequency (uncoupled)
ω_1, ω_2	characteristic yaw-roll frequencies (coupled)

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I. INTRODUCTION

It has been recognized for many years that a rigid body tends to align itself with its axis of minimum moment of inertia in the direction of the gradient of a gravitational field. Until relatively recent times, this effect was primarily of interest as an explanation of the fact that the moon maintains the same face toward the earth. However, with the development of artificial satellites, it was suggested (see Ref. 1) that this same effect could be used to stabilize such vehicles so that the orientation relative to the earth remains the same at all times. The feasibility of such a system for some applications has recently been demonstrated (see Ref. 2).

The use of the gravity gradient to achieve attitude stability has the attractive feature that the system is completely passive and thus puts no limitation on vehicle lifetime. However, since the stabilizing moments developed are dependent upon the nonuniformity of the gravitational field over the dimensions of the vehicle, it is obvious that these moments are extremely weak. Thus, a relatively small disturbing moment might override the inherent stability of the vehicle.

This Memorandum investigates the effect of several such sources of perturbations on the performance of a gravity gradient stabilization system. The perturbations considered include the effects of auxiliary rotating equipment, orbital eccentricity, as well as external moments due to such factors as radiation pressure, micro-meteoroid impact, and station-keeping propulsion.

II. ANALYSIS

STATEMENT OF THE PROBLEM

For the purposes of this Memorandum, it is assumed that the satellite vehicle is on a nearly circular orbit and its attitude is specified by the coordinate systems shown in Fig. 1. The X, Y, Z system is an orbital coordinate system with the Y axis along the instantaneous vertical and the Z axis along the normal to the orbital plane. The x, y, z system is a set of body axes fixed in the vehicle and with an orientation relative to X, Y, Z specified by the angles α , β and ψ in pitch, roll and yaw respectively. By a suitable choice of the moments of inertia in pitch, roll and yaw, the vehicle will be stable when α , β and ψ are all zero. The problem then is to determine how much this stability is disturbed by various perturbing effects.

GENERAL EQUATIONS OF MOTION

The general equations describing the motion of the satellite about its center of mass are derived in Ref. 3 and can be expressed as

$$\frac{d\omega_x}{dt} + \left(\frac{I_z - I_y}{I_x} \right) \omega_y \omega_z = \frac{3g_o R_o^2}{r^3} \left(\frac{I_z - I_y}{I_x} \right) b_y c_y + \frac{M_x}{I_x} \quad (1)$$

$$\frac{d\omega_y}{dt} + \left(\frac{I_x - I_z}{I_y} \right) \omega_z \omega_x = \frac{3g_o R_o^2}{r^3} \left(\frac{I_x - I_z}{I_y} \right) a_y c_y + \frac{M_y}{I_y} \quad (2)$$

$$\frac{d\omega_z}{dt} + \left(\frac{I_y - I_x}{I_z} \right) \omega_x \omega_y = \frac{3g_o R_o^2}{r^3} \left(\frac{I_y - I_x}{I_z} \right) b_y a_y + \frac{M_z}{I_z} \quad (3)$$

where ω_x , ω_y , ω_z are the components of the vehicle angular velocity relative to inertial space and are given by

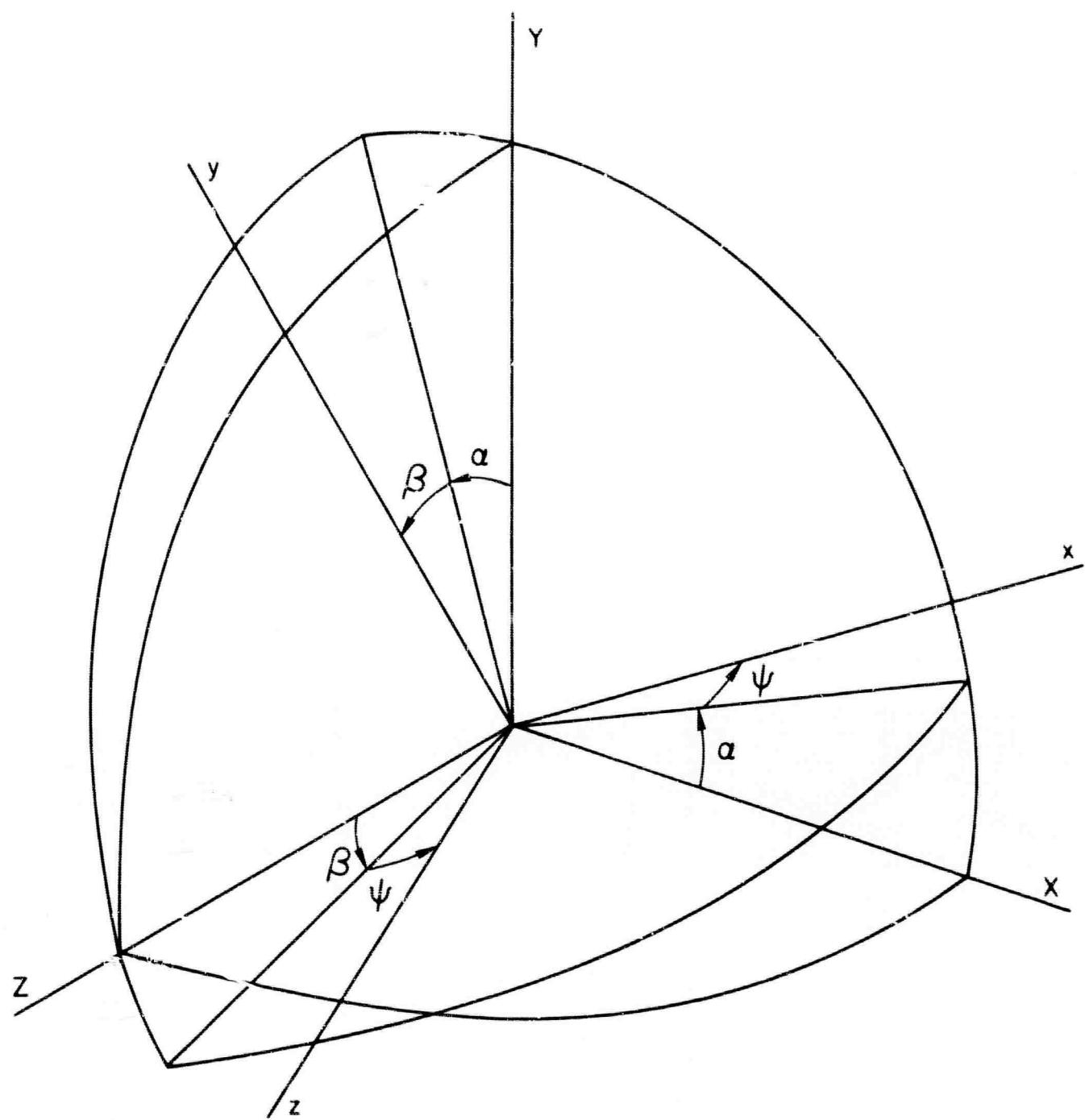


Fig.1—Coordinate systems

$$\omega_x = -(\dot{\alpha} + \dot{\theta}) \cos \beta \sin \psi + \dot{\beta} \cos \psi \quad (4)$$

$$\omega_y = (\dot{\alpha} + \dot{\theta}) \sin \beta + \dot{\psi} \quad (5)$$

$$\omega_z = (\dot{\alpha} + \dot{\theta}) \cos \beta \cos \psi + \dot{\beta} \sin \psi \quad (6)$$

while a_y , b_y , c_y are the direction cosines of the Y axis in the x, y, z system and are given by

$$a_y = \sin \alpha \cos \psi + \cos \alpha \sin \beta \sin \psi \quad (7)$$

$$b_y = \cos \alpha \cos \beta \quad (8)$$

$$c_y = \sin \alpha \sin \psi - \cos \alpha \sin \beta \cos \psi \quad (9)$$

The quantities I_x , I_y and I_z are the principal moments of inertia about the roll, yaw and pitch axes, while M_x , M_y and M_z are the corresponding external moments about these same axes. In addition, R_o is the radius of the earth, g_o is the gravitational acceleration at the earth's surface, r is the radial distance from the earth's center to the center of mass of the vehicle and $\dot{\theta}$ is the instantaneous orbital angular rate.

LINEARIZED EQUATIONS OF MOTION

Since in any satisfactory stabilization system large angular deviations in attitude cannot be tolerated, it is permissible to linearize the equations of motion developed above by the following substitutions

$$\omega_x = -\dot{\theta}\psi + \dot{\beta} \quad (10)$$

$$\omega_y = \dot{\theta}\beta + \dot{\psi} \quad (11)$$

$$\omega_z = \dot{\theta} + \dot{\alpha} \quad (12)$$

while

$$a_y = \alpha \quad (13)$$

$$b_y = 1 \quad (14)$$

$$c_y = -\beta \quad (15)$$

In addition, for a nearly circular orbit, the orbital angular rate is given by

$$\dot{\theta}^2 = \frac{g_o R_o^2}{r_c^3} = \omega_o^2 \quad (16)$$

Substitution of Eqs. (10)-(16) in Eqs. (1)-(3) gives

$$\ddot{\psi} + 4\omega_o^2 \left(\frac{I_z - I_y}{I_x} \right) \dot{\psi} + \left(\frac{I_z - I_y - I_x}{I_x} \right) \omega_o \dot{\psi} = \frac{M_x}{I_x} \quad (17)$$

$$\ddot{\psi} + \omega_o^2 \left(\frac{I_z - I_x}{I_y} \right) \dot{\psi} - \left(\frac{I_z - I_y - I_x}{I_y} \right) \omega_o \dot{\psi} = \frac{M_y}{I_y} \quad (18)$$

$$\ddot{\alpha} + 3\omega_o^2 \left(\frac{I_x - I_y}{I_z} \right) \dot{\alpha} = \frac{M_z}{I_z} \quad (19)$$

These equations can be further simplified by defining the following parameters

$$R_x = \frac{I_z - I_y}{I_x} \quad (20)$$

$$R_y = \frac{I_z - I_x}{I_y} \quad (21)$$

$$R_z = \frac{I_x - I_y}{I_z} \quad (22)$$

Substitution of Eqs. (20)-(22) into Eqs. (17)-(18) results in

$$\ddot{\beta} + 4R_x \omega_0^2 \beta + (R_x - 1) \omega_0 \dot{\psi} = \frac{M_x}{I_x} \quad (23)$$

$$\ddot{\psi} + R_y \omega_0^2 \psi - (R_y - 1) \omega_0 \dot{\beta} = \frac{M_y}{I_y} \quad (24)$$

$$\ddot{\alpha} + 3R_z \omega_0^2 \alpha = \frac{M_z}{I_z} \quad (25)$$

It should be noted that the quantities R_x , R_y and R_z are not independent and it is possible to express R_z in the form

$$R_z = \frac{R_x - R_y}{1 - R_x R_y} \quad (26)$$

In addition, there are certain other limitations on the values of R_x and R_y dictated by (1) the combinations of I_x , I_y and I_z which can be realized physically, and (2) the requirement that the system be inherently stable in pitch, yaw and roll. These limitations and the natural frequencies in pitch, yaw and roll are determined in the Appendix.

PERTURBATION EFFECTS

In this section the effect of various disturbances on a gravity gradient stabilized satellite vehicle will be considered. The disturbances included are (1) an arbitrary impulse moment such as might result from micro-meteoroid impact or from station-keeping propulsion, (2) a step increase in the external moment such as that due to radiation pressure or a steady-state thrust, (3) eccentricity of the orbit and (4) the interaction between the vehicle and on-board rotating equipment.

Impulse Moment

Pitch Excitation. If it is assumed that an impulsive moment of magnitude I_p is applied about the pitch or z axis, then the external moment M_z is given by

$$M_z = I_p \delta(0) \quad (27)$$

where $\delta(0)$ is a Dirac delta function at $t = 0$.

Equation (25) can now be written in the form

$$\ddot{\alpha} + \omega_{\alpha}^2 \alpha = \frac{I_p}{I_z} \delta(0) \quad (28)$$

from which the pitch response is obtained as

$$\alpha = \frac{I_p}{I_z \omega_{\alpha}} \sin \omega_{\alpha} t \quad (29)$$

Thus, the maximum pitch angle is given by

$$\alpha_m = \frac{I_p}{I_z \omega_{\alpha}} \quad (30)$$

which can be rewritten by means of Eqs. (16) and (191) in the form

$$\alpha_m = \frac{I_p}{I_z R_o} \sqrt{\frac{r_c^3 (1 - \frac{R_x R_y}{R_x^2})}{3g_o (R_x - R_y)}} \quad (31)$$

where r_c is the orbital radius.

Roll Excitation. If an impulsive moment is applied about the roll or x axis, the value of M_x is given by

$$M_x = I_R \delta(0) \quad (32)$$

Since the roll and yaw motions are coupled, it is necessary to solve Eqs. (23) and (24) simultaneously. These equations can be

rewritten in the form

$$\ddot{\beta} + \omega_B^2 \beta + \mu_B \dot{\psi} = \frac{I_R}{I_X} \delta(0) \quad (33)$$

$$- \mu_\psi \dot{\beta} + \ddot{\psi} + \omega_\psi^2 \psi = 0 \quad (34)$$

where

$$\omega_B^2 = 4R_X \omega_0^2 \quad (35)$$

$$\omega_\psi^2 = R_Y \omega_0^2 \quad (36)$$

$$\mu_B = (R_X - 1) \omega_0 \quad (37)$$

$$\mu_\psi = (R_Y - 1) \omega_0 \quad (38)$$

The solution to Eqs. (33) and (34) is given by

$$\beta = \frac{I_R}{I_X(\omega_1^2 - \omega_2^2)} \left[\frac{\omega_1^2 - \omega_\psi^2}{\omega_1} \sin \omega_1 t + \frac{\omega_\psi^2 - \omega_2^2}{\omega_2} \sin \omega_2 t \right] \quad (39)$$

and

$$\psi = - \frac{I_R \mu_\psi}{I_X(\omega_1^2 - \omega_2^2)} \left[\cos \omega_1 t - \cos \omega_2 t \right] \quad (40)$$

where ω_1 and ω_2 are the two characteristic frequencies of the yaw-roll response, as determined in the Appendix.

Since both the roll and pitch responses are made up of the sum of two sinusoidal terms of frequencies, ω_1 and ω_2 , the maximum possible value would be the sum of the amplitudes of the individual terms.

Thus,

$$\beta_m = \frac{I_R}{I_x(\omega_1^2 - \omega_2^2)} \left[\frac{\omega_1^2 - \omega_\beta^2}{\omega_1} + \frac{\omega_\beta^2 - \omega_2^2}{\omega_2} \right] = \frac{I_R}{I_x} \frac{(\omega_\beta^2 + \omega_1 \omega_2)}{\omega_1 \omega_2 (\omega_1 + \omega_2)} \quad (41)$$

while the associated yaw response has a maximum amplitude of

$$\psi_m = - \frac{2I_R \mu_\psi}{I_x(\omega_1^2 - \omega_2^2)} \quad (42)$$

It should be noted that Eq. (41) is valid only if R_x and R_y are both positive, which is the case for most configurations considered for gravity gradient stabilization. If R_x and R_y were negative, the second fraction in the brackets in Eq. (41) would have a negative sign.

Yaw Excitation. If an impulsive moment is applied about the yaw or y axis, the resulting value of M_y is given by

$$M_y = I_y \delta(0) \quad (43)$$

and Eqs. (23) and (24) have the form

$$\ddot{\beta} + \omega_\beta^2 \beta + \mu_\beta \dot{\psi} = 0 \quad (44)$$

$$-\mu_\psi \dot{\beta} + \ddot{\psi} + \omega_\psi^2 \psi = \frac{I_y}{I_y} \delta(0) \quad (45)$$

The solution for the resulting yaw and roll responses is given by the relations

$$\psi = \frac{I_y}{I_y(\omega_1^2 - \omega_2^2)} \left[\frac{\omega_1^2 - \omega_\beta^2}{\omega_1} \sin \omega_1 t + \frac{\omega_\beta^2 - \omega_2^2}{\omega_2} \sin \omega_2 t \right] \quad (46)$$

and

$$\theta = \frac{I_Y \mu_B}{I_y(\omega_1^2 - \omega_2^2)} \left[\cos \omega_1 t - \cos \omega_2 t \right] \quad (47)$$

As in the previous case, the maximum possible amplitudes in yaw and roll are given by

$$\psi_m = \frac{I_Y}{I_y(\omega_1^2 - \omega_2^2)} \left[\frac{\omega_1^2 - \omega_B^2}{\omega_1} + \frac{\omega_B^2 - \omega_2^2}{\omega_2} \right] = \frac{I_Y (\omega_B^2 + \omega_1 \omega_2)}{I_y \omega_1 \omega_2 (\omega_1 + \omega_2)} \quad (48)$$

and

$$\beta_m = - \frac{2I_Y \mu_B}{I_y(\omega_1^2 - \omega_2^2)} \quad (49)$$

As in the case for roll excitation, Eq. (48) applies only when R_x and R_y are positive. If R_x and R_y are negative, the sign of the second fraction in brackets reverses.

It should be noted that in this and the succeeding analyses, no damping has been incorporated in the control system. If this were included, the transients resulting from impulse moments would eventually damp out. However, the expressions given for the maximum values of α , β and ψ are a good measure of the attitude disturbances which might exist during the early part of the transient response since the time constant of a system with damping is of the order of several orbital periods.

Step Moment

Pitch Excitation. If a step change occurs in the moment applied about the pitch axis such that

$$\begin{aligned} M_z &= 0 & (t < 0) \\ &= M_p & (t > 0) \end{aligned} \quad (50)$$

then Eq. (25) can be written as

$$\ddot{\alpha} + \omega_{\alpha}^2 \alpha = \frac{M_p}{I_z} \quad (51)$$

which has a solution given by

$$\alpha = \frac{M_p}{I_z \omega_{\alpha}^2} (1 - \cos \omega_{\alpha} t) \quad (52)$$

It is seen that, in the absence of damping, the pitch angle reaches a maximum of

$$\begin{aligned} \alpha_m &= \frac{2M_p}{\omega_{\alpha}^2 I_z} \\ &= \frac{2M_p r_c^3 (1 - R_x R_y)}{3I_z R_o^2 g_o (R_x - R_y)} \end{aligned} \quad (53)$$

If damping is present, then the steady-state value of the pitch angle is half that shown above or

$$\begin{aligned} \alpha_{ss} &= \frac{M_p}{\omega_{\alpha}^2 I_z} \\ &= \frac{M_p r_c^3 (1 - R_x R_y)}{3I_z R_o^2 g_o (R_x - R_y)} \end{aligned} \quad (54)$$

Roll Excitation. If a step change occurs in the moment applied about the roll axis, Eqs. (23) and (24) representing the coupled yaw-roll response can be written as

$$\ddot{\beta} + \omega_{\beta}^2 \beta + \mu_{\beta} \dot{\psi} = \frac{M_R}{I_x} \quad (55)$$

$$- \mu_{\psi} \dot{\beta} + \ddot{\psi} + \omega_{\psi}^2 \psi = 0 \quad (56)$$

The resulting yaw and roll responses are given by

$$\theta = \frac{M_R}{I_x} \left[\frac{\omega_\psi^2}{\omega_1^2 - \omega_2^2} - \frac{\omega_1^2 - \omega_\psi^2}{\omega_1^2(\omega_1^2 - \omega_2^2)} \cos \omega_1 t - \frac{\omega_\psi^2 - \omega_2^2}{\omega_2^2(\omega_1^2 - \omega_2^2)} \cos \omega_2 t \right] \quad (57)$$

and

$$\psi = \frac{M_R \omega_\psi}{I_x} \left[-\frac{1}{\omega_1^2(\omega_1^2 - \omega_2^2)} \sin \omega_1 t + \frac{1}{\omega_2^2(\omega_1^2 - \omega_2^2)} \sin \omega_2 t \right] \quad (58)$$

An examination of Eqs. (57) and (58), together with Eqs. (21), (36), (193) and (194), shows that the oscillatory components of the responses have maximum possible amplitudes given by

$$\begin{aligned} \theta_m &= \frac{M_R}{I_x} \left[\frac{\omega_1^2 - \omega_\psi^2}{\omega_1^2(\omega_1^2 - \omega_2^2)} + \frac{\omega_\psi^2 - \omega_2^2}{\omega_2^2(\omega_1^2 - \omega_2^2)} \right] \\ &= \frac{M_R \omega_\psi^2}{I_x \omega_1^2 \omega_2^2} \\ &= \frac{M_R}{4(I_z - I_y) \omega_o^2} \end{aligned} \quad (59)$$

and

$$\psi_m = \frac{M_R \omega_\psi}{I_x(\omega_1^2 - \omega_2^2)} \left[\frac{1}{\omega_1} + \frac{1}{\omega_2} \right] = \frac{M_R (R_y - 1) \omega_o}{I_x \omega_1 \omega_2 (\omega_1^2 - \omega_2^2)} \quad (60)$$

In addition to the transient terms discussed above, the roll response, Eq. (57), also has a steady-state term given by the expression

$$\begin{aligned} \theta_{ss} &= \frac{M_R \omega_\psi^2}{I_x \omega_1^2 \omega_2^2} \\ &= \frac{M_R}{4(I_z - I_y) \omega_o^2} \end{aligned}$$

$$= \frac{M_R r_c^3}{4g_o R_o^2 (I_z - I_y)} \quad (61)$$

It is seen that β_m and β_{ss} are equal; thus, as a worst case the roll angle might have a value equal to twice that given in Eq. (61).

Yaw Excitation. If a step change in the yaw moment occurs, Eqs. (23) and (24) take the form

$$\ddot{\beta} + \omega_B^2 \beta + \mu_B \dot{\psi} = 0 \quad (62)$$

$$- \mu_\psi \dot{\beta} + \ddot{\psi} + \omega_\psi^2 \psi = \frac{M_Y}{I_y} \quad (63)$$

By analogy with the previous case, the yaw and roll responses can be written as

$$\psi = \frac{M_Y}{I_y} \left[\frac{\omega_B^2}{\omega_1^2 - \omega_2^2} - \frac{\omega_1^2 - \omega_2^2}{\omega_1(\omega_1^2 - \omega_2^2)} \cos \omega_1 t - \frac{\omega_1^2 - \omega_2^2}{\omega_2(\omega_1^2 - \omega_2^2)} \cos \omega_2 t \right] \quad (64)$$

and

$$\beta = \frac{M_Y \mu_B}{I_y} \left[\frac{1}{\omega_1(\omega_1^2 - \omega_2^2)} \sin \omega_1 t - \frac{1}{\omega_2(\omega_1^2 - \omega_2^2)} \sin \omega_2 t \right] \quad (65)$$

The maximum possible amplitudes of the oscillatory terms in Eqs. (64) and (65) are determined as in the previous case, and are of the form

$$\begin{aligned} \psi_m &= \frac{M_Y \omega_B^2}{I_y \omega_1^2 \omega_2^2} \\ &= \frac{M_Y}{(I_z - I_x) \omega_0^2} \end{aligned} \quad (66)$$

and

$$\beta_m = \frac{M_Y \mu_B}{I_y (\omega_1^2 - \omega_2^2)} \left[\frac{1}{\omega_1} + \frac{1}{\omega_2} \right] = \frac{M_Y (R_x - 1) \omega_0}{I_y \omega_1 \omega_2 (\omega_1 - \omega_2)} \quad (67)$$

In this case, there is a steady-state deflection in yaw given by

$$\begin{aligned} \psi_{ss} &= \frac{M_Y \mu_B^2}{I_y \omega_1^2 \omega_2^2} \\ &= \frac{M_Y}{(I_z - I_x) \omega_0^2} \\ &= \frac{M_Y r_c^3}{g_o R_o^2 (I_z - I_x)} \end{aligned} \quad (68)$$

Again, the values of ψ_m and ψ_{ss} are equal and a yaw angle equal to their sum might result.

It should also be noted that if the pitch and roll moments of inertia are nearly equal, the resulting value of ψ_{ss} may be large.

Orbital Eccentricity

In the development of the linearized equations of motion, it has been assumed that the orbital angular rate $\dot{\theta}$ is a constant, ω_0 . However, if the orbit has a small eccentricity, $\dot{\theta}$ is no longer constant and the linearized equations obtained by the substitution of Eqs. (10)-(15) into Eqs. (1)-(3) have the form

$$\ddot{\theta} + R_x \dot{\theta}^2 \beta + (R_x - 1) \dot{\theta} \dot{\psi} = - \frac{3 g_o R_o^2}{r^3} R_x \beta \quad (69)$$

$$\ddot{\psi} + R_y \dot{\theta}^2 \psi - (R_y - 1) \dot{\theta} \dot{\beta} = 0 \quad (70)$$

$$\ddot{\theta} + \ddot{\alpha} = - \frac{3 g_o R_o^2}{r^3} R_z \alpha \quad (71)$$

The orbital motion of the satellite is specified by the relations

$$\frac{1}{r} = \frac{g_o R_o^2}{C_o^2} (1 + \epsilon \cos \theta) \quad (72)$$

$$r^2 \dot{\theta} = C_o \quad (73)$$

where C_o is a measure of the orbital angular momentum, ϵ is the orbital eccentricity and the angle θ is measured from perigee.

Pitch Response. By means of Eq. (73), it is possible to change the independent variable of Eq. (71) from t to θ so that

$$\frac{d^2 \alpha}{d\theta^2} - \frac{2}{r} \frac{dr}{d\theta} \left(1 + \frac{d\alpha}{d\theta} \right) + \frac{3 g_o R_o^2}{C_o^2} R_z r \alpha = 0 \quad (74)$$

Elimination of r between Eqs. (72) and (74) gives

$$\frac{d^2 \alpha}{d\theta^2} + 3R_z \alpha = 2\epsilon \sin \theta \quad (75)$$

where second-order terms have been neglected.

It is seen that the eccentricity appears as the amplitude of a sinusoidal driving function in Eq. (75) and the resulting pitch response is given by the relation

$$\alpha = \frac{2\epsilon}{3R_z - 1} \left[\sin \theta - \frac{1}{\sqrt{3R_z}} \sin \sqrt{3R_z} \theta \right] \quad (76)$$

The maximum possible pitch angle is expressed as

$$\alpha_m = \frac{2\epsilon}{3 \frac{z}{z-1}} \left[1 + \frac{1}{\sqrt{3R_z}} \right]$$

$$= \frac{2\epsilon}{\sqrt{3R_z} (\sqrt{3R_z} - 1)} \quad (77)$$

If the system has damping, the second term of Eq. (76) will eventually damp out and the resulting pitch amplitude will be given as

$$\alpha_m = \frac{2\epsilon}{3R_z - 1} \quad (78)$$

An examination of Eq. (76) shows that this solution is not valid if R_z is equal to either zero or one-third.

The first of these cases ($R_z = 0$) corresponds to a configuration which is symmetrical about the pitch axis and thus has no pitch stability. In the limit as R_z approaches zero, Eq. (76) reduces to

$$\alpha = 2\epsilon [\theta - \sin \theta] \quad (79)$$

Since this solution has a secular term, sooner or later it would violate the assumption that α is a small angle. In Fig. 3 of the Appendix, the locus for this case is the contour for ω_α equal to zero.

In the second case ($R_z = 1/3$) the term on the right in Eq. (75) is driving the system at resonance, and as R_z approaches 1/3, Eq. (76) takes the form

$$\alpha = \epsilon [\sin \theta - \theta \cos \theta] \quad (80)$$

This constitutes a divergent oscillation which would, in time, exceed the small angle assumption for α . This case corresponds to the contour

for ω_a/ω_o equal to unity in Fig. 3 of the Appendix.

Yaw-Roll Response. If Eqs. (69) and (70) are expressed with θ as the independent variable and second-order terms are neglected as in the pitch case, the resulting equations have the form

$$\frac{d^2\beta}{d\theta^2} + 4R_x\beta + (R_x - 1) \frac{d\psi}{d\theta} = 0 \quad (81)$$

$$\frac{d^2\psi}{d\theta^2} + R_y\psi - (R_y - 1) \frac{d\beta}{d\theta} = 0 \quad (82)$$

It is seen that no first-order terms involving the eccentricity are present and thus, no yaw-roll perturbation is caused by eccentricity.

Rotating Equipment

If rotating equipment is included in the payload of a satellite, the vehicle can no longer be treated as a single rigid body, but must be considered as a system of coupled bodies, consisting of the main body of the vehicle and the various internal moving parts. Since angular momentum must be conserved, it is obvious that the motion of these internal parts can react on the main satellite body to produce disturbances in its attitude.

This analysis is restricted to the case of a single rotating element on-board the satellite. It is assumed that the element has rotational symmetry about its spin axis and that this axis lies along one of the principal axes of the main satellite body. Under these conditions, the motion of the main body is specified by Eqs. (17)-(19) where M_x , M_y and M_z represent the moments applied to the main body by the rotating element either through its bearings or through its motor drive.

Pitch Excitation. If the rotating element has its spin axis along the pitch axis of the main vehicle, its equations of motion are given by

$$I_T \frac{d\omega_x}{dt} + I_s \omega_z \omega_y - I_T \omega_y \omega_z = 3\omega_0^2 (I_s - I_T) \dot{\theta}_x - M_x \quad (83)$$

$$I_T \frac{d\omega_y}{dt} + I_T \omega_x \omega_z - I_s \omega_z \omega_x = 3\omega_0^2 (I_T - I_s) c_y a_y - M_y \quad (84)$$

$$I_s \frac{d\omega_z}{dt} = - M_z \quad (85)$$

where I_s is the moment of inertia of the rotating element about its spin axis and I_T is the value for the two transverse axes. The quantity ω_{z2} is the total angular velocity of the rotor about its spin axis and is given by

$$\omega_{z2} = \omega_z + \dot{\alpha}_2 \quad (86)$$

where $\dot{\alpha}_2$ is the spin rate of the rotor relative to the main body.

Substitution of Eqs. (10)-(15) and Eq. (86) into Eqs. (83)-(85) yields the following small angle equations for the motion of the rotor

$$I_T \ddot{\theta} + 4\omega_0^2 (I_s - I_T) \theta + \omega_0 \dot{\psi} (I_s - 2I_T) + I_s \dot{\alpha}_2 (\dot{\theta} + \omega_0 \dot{\psi}) = - M_x \quad (87)$$

$$I_T \ddot{\psi} + \omega_0^2 (I_s - I_T) \dot{\psi} - \omega_0 \dot{\theta} (I_s - 2I_T) - I_s \dot{\alpha}_2 (\dot{\theta} - \omega_0 \dot{\psi}) = - M_y \quad (88)$$

$$I_s (\ddot{\alpha}_2 + \ddot{\alpha}_2) = - M_z \quad (89)$$

Since there is no differential rotation of the two bodies about the x or y axes, the quantities M_x and M_y represent constraint moments applied through the bearings normal to the axis of rotation. Thus,

M_x and M_y can be eliminated by adding Eqs. (17) and (87) as well as Eqs. (18) and (88) to give

$$\ddot{\beta} + 4\omega_0^2 \left(\frac{I_z' - I_y'}{I_x'} \right) \beta + \left(\frac{I_z' - I_y' - I_x'}{I_x'} \right) \omega_0 \dot{\psi} + I_s \dot{\alpha}_2 (\dot{\psi} + \omega_0 \beta) = 0 \quad (90)$$

and

$$\ddot{\psi} + \omega_0^2 \left(\frac{I_z' - I_x'}{I_y'} \right) \psi - \left(\frac{I_z' - I_y' - I_x'}{I_y'} \right) \omega_0 \dot{\beta} - I_s \dot{\alpha}_2 (\dot{\beta} - \omega_0 \psi) = 0 \quad (91)$$

where

$$I_x' = I_x + I_T$$

$$I_y' = I_y + I_T \quad (92)$$

$$I_z' = I_z + I_s$$

An examination of Eqs. (90) and (91) shows that the effect of $\dot{\alpha}_2$ is to increase the yaw-roll coupling as well as the yaw and roll stiffness. However, if the initial conditions for the yaw and roll angles ψ and β are zero, the angular rate $\dot{\alpha}_2$ produces no disturbance in either yaw or roll.

Thus, the effect of $\dot{\alpha}_2$ is to cause variations in the pitch attitude of the main body as described by Eqs. (19) and (89). In this case the moment M_z is exerted through the motor drive on the two bodies. In order to solve Eqs. (19) and (89), it is necessary to obtain an expression for M_z as a function of $\dot{\alpha}_2$. This is done with a simple motor equation of the form

$$-M_z = k_c i - f\dot{\alpha}_2 \quad (93)$$

where $-M_z$, the moment applied to the rotor, is made up of a term proportional to the current i , and a viscous drag term proportional to $\dot{\alpha}_2$.

The motor current can be expressed by the relation

$$E = R_i + k_1 \dot{\alpha}_2 \quad (94)$$

where the applied voltage, E , is equal to the drop across the motor resistance plus a back emf term proportional to $\dot{\alpha}_2$.

Substitution of Eqs. (93) and (94) into Eqs. (19) and (89) gives

$$\ddot{\alpha} + \omega_{\alpha}^2 \alpha - \frac{F}{I_z} \dot{\alpha}_2 = - \frac{k_o}{I_z R} E \quad (95)$$

$$\ddot{\alpha} + \ddot{\alpha}_2 + \frac{F}{I_s} \dot{\alpha}_2 = \frac{k_o}{I_s R} E \quad (96)$$

where

$$F = f + \frac{k_1 k_o}{R} \quad (97)$$

It can be shown that if $I_z \gg I_s$, then the solution for Eqs. (95) and (96) for a step change in motor voltage is given by

$$\alpha = - \frac{k_o E T_o^2}{R I_z (\omega_{\alpha}^2 T_o^2 + 1)} \left[e^{-\frac{t}{T_o}} - \cos \omega_{\alpha} t + \frac{1}{\omega_{\alpha} T_o} \sin \omega_{\alpha} t \right] \quad (98)$$

and

$$\dot{\alpha}_2 = \frac{k_o E T_o^2}{R I_s} \left(1 - e^{-\frac{t}{T_o}} \right) \quad (99)$$

where

$$T_o = \frac{I_s}{F} \quad (100)$$

and is the time required for the motor to come up to its steady speed.

The value of this steady speed, $\dot{\alpha}_{2ss}$, is determined from Eq. (99) as

$$\dot{\alpha}_{2ss} = \frac{k_o ET_o}{KI_s} \quad (101)$$

and Eqs. (98) and (99) can be put in the form

$$\alpha = - \frac{\dot{\alpha}_{2ss} I_s T_o}{I_z (1 + \omega_\alpha^2 T_o^2)} \left[e^{-\frac{t}{T_o}} - \cos \omega_\alpha t + \frac{1}{\omega_\alpha T_o} \sin \omega_\alpha t \right] \quad (102)$$

$$\dot{\alpha}_2 = \dot{\alpha}_{2ss} \left(1 - e^{-\frac{t}{T_o}} \right) \quad (103)$$

If the time constant T_o is small compared to the orbital period, then

$$\omega_\alpha T_o \ll 1$$

and Eq. (102) can be written as

$$\alpha = - \frac{\dot{\alpha}_{2ss} I_s}{\omega_\alpha I_z} \sin \omega_\alpha t \quad (104)$$

Thus, if the voltage applied to the motor is increased by a step, an oscillation in the pitch attitude of the main body will result, as indicated in Eq. (104).

Yaw Excitation. If the rotating element has its spin axis along the yaw axis of the main vehicle, its equations of motion are given by

$$I_T \frac{d\omega_x}{dt} + I_I \omega_z \omega_y - I_s \omega_y^2 \omega_z = 3\omega_0^2 (I_T - I_s) b_y c_y - M_x \quad (105)$$

$$I_s \frac{d\omega_y^2}{dt} = - M_y \quad (106)$$

$$I_T \frac{d\omega_z}{dt} + I_s \omega_y 2\omega_x - I_T \omega_x \omega_y = 3\omega_o^2 (I_s - I_T) a_y b_y - M_z \quad (107)$$

As before, I_s and I_T are the spin and transverse moments of inertia respectively, and $\omega_y 2$ is the total angular velocity of the rotor about its spin axis given by

$$\omega_y 2 = \omega_y + \dot{\psi}_2 \quad (108)$$

Substitution of Eqs. (10)-(15) and Eq. (108) in Eqs. (105)-(107) gives the small angle form as

$$I_T \ddot{\beta} + 4\omega_o^2 (I_T - I_s) \beta - I_s \omega_o \dot{\psi} - I_s \omega_o \dot{\psi}_2 = -M_x \quad (109)$$

$$I_s \ddot{\psi} + I_s \omega_o \dot{\beta} + I_s \ddot{\psi}_2 = -M_y \quad (110)$$

$$I_T \ddot{\alpha} + 3\omega_o^2 (I_T - I_s) \alpha + I_s \dot{\psi}_2 (\dot{\beta} - \omega_o \psi) = -M_z \quad (111)$$

where it is assumed that $\dot{\alpha} \ll \omega_o$. In this case, M_x and M_z are constraint moments and can be eliminated between Eqs. (17) and (109) and Eqs. (19) and (111), respectively, to give

$$\ddot{\beta} + 4\omega_o^2 \left(\frac{I'_z - I'_y}{I'_x} \right) \beta + \left(\frac{I'_z - I'_y - I'_x}{I'_x} \right) \omega_o \dot{\psi} - \frac{I_s \omega_o \dot{\psi}_2}{I'_x} = 0 \quad (112)$$

$$\ddot{\alpha} + 3\omega_o^2 \left(\frac{I'_x - I'_y}{I'_z} \right) \alpha + \frac{I_s \dot{\psi}_2}{I'_z} (\dot{\beta} - \omega_o \psi) = 0 \quad (113)$$

where

$$I'_x = I_x + I_T$$

$$I'_y = I_y + I_s \quad (114)$$

$$I'_z = I_z + I_T$$

An examination of Eq. (113) shows that the term involving $\dot{\psi}_2$ is negligible. Thus, the pitch behavior of the vehicle is essentially independent of internal rotations about the yaw axis.

It is now necessary to obtain an expression for the interaction moment M_y . This is done in the same manner as in the pitch case with the result that

$$M_y = F\dot{\psi}_2 - \frac{k_o}{R} E \quad (115)$$

The yaw-roll behavior of the vehicle is now specified by Eqs. (18), (110), (111) and (115), which can be combined to give the following

$$\ddot{\beta} + \omega_B^2 \beta + \mu_B \dot{\beta} - \frac{I_s \omega_o}{I'_x} \dot{\psi}_2 = 0 \quad (116)$$

$$\ddot{\psi} + \omega_B^2 \psi - \mu_B \dot{\beta} + \frac{I_s}{I'_y} \ddot{\psi}_2 = 0 \quad (117)$$

$$\ddot{\psi} + \omega_o^2 \beta + \ddot{\psi}_2 + \frac{1}{T_o} \dot{\psi}_2 = \frac{k_o E}{I_s R} \quad (118)$$

where the definitions of ω_B , ω_ψ , μ_B and μ_ψ are of the same form as Eqs. (35)-(38), using the primed values of the moments of inertia to determine R_x and R_y .

It can be shown that if T_c is small compared to the orbital period, then $\omega_1 T_o$ and $\omega_2 T_o$ are small compared to unity since from the Appendix it is seen that ω_1 and ω_2 are of the same order as ω_o . If, in addition, I_s is small compared to both I'_x and I'_y , then a good approximation to the solution of Eqs. (116)-(118) for a step change in E is given by

$$\dot{\psi}_2 = \dot{\psi}_{2ss} \left(\frac{1 - e^{-\frac{t}{T_o}}}{1 - e^{-\frac{t}{T_o}}} \right) \quad (119)$$

$$B = \frac{\dot{\psi}_{2ss} I_s \omega_0 (I'_z - I'_x)}{I'_x I'_y} \left[\frac{\omega_0^2}{\frac{2}{\omega_1^2} - T_0^2} e^{-\frac{t}{T_0}} - \frac{\omega_1^2 - \omega_c^2}{\omega_1^2 (\omega_1^2 - \omega_2^2)} \cos \omega_1 t \right. \\ \left. - \frac{\omega_0^2 - \omega_2^2}{\omega_2^2 (\omega_1^2 - \omega_2^2)} \cos \omega_2 t \right] \quad (120)$$

$$\dot{\psi} = - \frac{\dot{\psi}_{2ss} I_s}{I'_y} \left[T_0 e^{-\frac{t}{T_0}} + \frac{\omega_1^2 - \omega_c^2}{\omega_1 (\omega_1^2 - \omega_2^2)} \sin \omega_1 t \right. \\ \left. + \frac{\omega_c^2 - \omega_2^2}{\omega_2 (\omega_1^2 - \omega_2^2)} \sin \omega_2 t \right] \quad (121)$$

where the steady-state angular rate of the motor, $\dot{\psi}_{2ss}$, is given by

$$\dot{\psi}_{2ss} = \frac{k_o T_o E}{I_s R} \quad (122)$$

and

$$\omega_c'^2 = \omega_0^2 (1 + 3R_x) \quad (123)$$

The values of ω_1 and ω_2 are determined as described in the Appendix, using the primed values of the moments of inertia to determine R_x and R_y .

It is seen that the exponential terms in Eqs. (119)-(121) become negligible after a relatively short time (of the order of a few time constants). Thus, the roll response, Eq. (120), is made up of a steady-state term and two oscillatory terms of frequencies ω_1 and ω_2 . The amplitude of the steady-state term is given by

$$\theta_{ss} = \frac{\dot{\psi}_{2ss} I_s \omega_0^3 (I_z' - I_x')}{I_x' I_y' \omega_1^2 \omega_2^2} = \frac{\dot{\psi}_{2ss} I_s}{4\omega_0 (I_z' - I_y')} \quad (124)$$

The oscillatory terms in the roll response have a maximum possible amplitude equal to the sum of their individual amplitudes given by

$$\begin{aligned} \theta_m &= \frac{\dot{\psi}_{2ss} I_s \omega_0 (I_z' - I_x')}{I_x' I_y'} \left[\frac{\omega_1^2 - \omega_0^2}{\omega_1^2 (\omega_1^2 - \omega_2^2)} + \frac{\omega_0^2 - \omega_2^2}{\omega_2^2 (\omega_1^2 - \omega_2^2)} \right] \\ &= \frac{\dot{\psi}_{2ss} I_s}{4\omega_0 (I_z' - I_y')} \quad (125) \end{aligned}$$

This amplitude is equal to the steady-state value determined in Eq. (124). Thus, the maximum roll angle would be twice that indicated in Eqs. (124) or (125). However, in the presence of damping the oscillatory amplitude would reduce to zero while the steady-state value would remain unaffected.

The yaw response consists of two oscillatory terms with a maximum possible amplitude given by

$$\begin{aligned} \psi_m &= \frac{\dot{\psi}_{2ss} I_s}{I_y'} \left[\frac{\omega_1^2 - \omega_0'^2}{\omega_1^2 (\omega_1^2 - \omega_2^2)} + \frac{\omega_0'^2 - \omega_2^2}{\omega_2^2 (\omega_1^2 - \omega_2^2)} \right] \\ &= \frac{\dot{\psi}_{2ss} I_s [\omega_1 \omega_2 + \omega_0^2 (1 + 3R_x)]}{I_y' \omega_1 \omega_2 (\omega_1 + \omega_2)} \quad (126) \end{aligned}$$

Again, it should be emphasized that a damping system would reduce this yaw oscillation to zero.

Roll Excitation. When the rotating element has its spin axis along the roll axis of the main vehicle, the equations of motion are developed in a completely analogous manner to that used in the case of yaw excitation. The resulting pitch equation has the form

$$\ddot{\alpha} + 3\omega_0^2 \left(\frac{I'_x - I'_y}{I'_z} \right) \alpha - \frac{I_s}{I'_z} \dot{\beta}_2 (\dot{\psi} + \omega_0 \beta) = 0 \quad (127)$$

where

$$I'_x = I_x + I_s$$

$$I'_y = I_y + I_T \quad (128)$$

$$I'_z = I_z + I_T$$

As is the yaw case above, the term in Eq. (127) involving the angular rate of the motor, $\dot{\beta}_2$, is negligible, and the pitch equation is essentially independent of rotations about the roll axis.

The equations describing the motion of the vehicle in yaw and roll and the coupled motion of the motor have the form

$$\ddot{\psi} + \omega_\psi^2 \psi - \mu_\psi \dot{\beta} + \frac{I_s}{I'_y} \omega_0 \dot{\beta}_2 = 0 \quad (129)$$

$$\mu_\beta \dot{\psi} + \ddot{\beta} + \omega_\beta^2 \beta + \frac{I_s}{I'_y} \ddot{\beta}_2 = 0 \quad (130)$$

$$- \omega_0 \dot{\psi} + \ddot{\beta} + \ddot{\beta}_2 + \frac{1}{T_o} \dot{\beta}_2 = \frac{k_o E}{I_s R} \quad (131)$$

under the assumption that $\dot{\alpha} \ll \omega_0$.

If, as before, $\omega_1 T_o$ and $\omega_2 T_o$ are much less than unity and I_s is much less than I'_x and I'_y , then the solution to Eqs. (129)-(131) for a step in E is given by

$$\dot{\beta}_2 = \dot{\beta}_{2ss} \left(1 - e^{-\frac{t}{T_o}} \right) \quad (132)$$

$$\beta = - \frac{\dot{\beta}_{2ss} I_s}{I'_x} \left[T_o e^{-\frac{t}{T_o}} + \frac{(\omega_1^2 - \omega_0^2)}{\omega_1(\omega_1^2 - \omega_2^2)} \sin \omega_1 t + \frac{\omega_0^2 - \omega_2^2}{\omega_2(\omega_1^2 - \omega_2^2)} \sin \omega_2 t \right] \quad (133)$$

$$\psi = - \frac{\dot{\beta}_{2ss} I_s \omega_0 (I'_z - I'_y)}{I'_x I'_y} \left[\frac{4\omega_0^2}{\omega_1^2 \omega_2^2} e^{-\frac{t}{T_o}} + \frac{4\omega_0^2 - \omega_1^2}{\omega_1^2 (\omega_1^2 - \omega_2^2)} \cos \omega_1 t - \frac{4\omega_0^2 - \omega_2^2}{\omega_2^2 (\omega_1^2 - \omega_2^2)} \cos \omega_2 t \right] \quad (134)$$

where

$$\dot{\beta}_{2ss} = \frac{k_o T_o E}{I_s R} \quad (135)$$

As before, the exponential terms in Eqs. (133) and (134) can be neglected after a short time, and it is seen that the steady-state yaw response is given by

$$\psi_{ss} = - \frac{4\dot{\beta}_{2ss} I_s \omega_0^3 (I'_z - I'_y)}{I'_x I'_y \omega_1^2 \omega_2^2} \quad (136)$$

or

$$\psi_{ss} = - \frac{\dot{\beta}_{2ss} I_s}{\omega_0 (I'_z - I'_x)} \quad (137)$$

The maximum possible amplitude of the oscillatory part of the yaw response is the sum of the absolute values of the amplitudes of the last two terms of Eq. (134) expressed in the form

$$\begin{aligned}\psi_m &= \frac{\dot{\beta}_{2ss} I_s \omega_0 (I'_z - I'_y)}{I'_x I'_y (\omega_1^2 - \omega_2^2)} \left[\frac{4\omega_0^2 - \omega_1^2}{\omega_1^2} + \frac{4\omega_0^2 - \omega_2^2}{\omega_2^2} \right] \\ &= \frac{\dot{\beta}_{2ss} I_s}{(I'_z - I'_x) \omega_0} \left[\frac{2\omega_0^2(\omega_1^2 + \omega_2^2) - \omega_1^2 \omega_2^2}{2\omega_0^2(\omega_1^2 - \omega_2^2)} \right] \quad (138)\end{aligned}$$

In this case it is seen that ψ_m is greater than ψ_{ss} .

The maximum possible amplitude of the oscillatory roll response is given by

$$\beta_m = \frac{\dot{\beta}_{2ss} I_s}{I'_x} \left[\frac{\omega_1^2 - \omega_0^2}{\omega_1^2(\omega_1^2 - \omega_2^2)} + \frac{\omega_0^2 - \omega_2^2}{\omega_2^2(\omega_1^2 - \omega_2^2)} \right] = \frac{\dot{\beta}_{2ss} I_s (\omega_0^2 + \omega_1 \omega_2)}{I'_x \omega_1 \omega_2 (\omega_1 + \omega_2)} \quad (139)$$

III. RESULTS

In this section, some typical values of the attitude perturbations of a satellite vehicle are determined by means of the formulas developed in the previous section.

ASSUMED VEHICLE CONFIGURATION

While it is not the purpose of this Memorandum to design a satellite vehicle, it is necessary to specify the three principal moments of inertia of the vehicle before the formulas of the previous section can be evaluated. Thus, for a nominal design the moments of inertia are specified as

$$\begin{aligned} I_x &= 9500 \text{ slug ft}^2 \\ I_y &= 1000 \text{ slug ft}^2 \\ I_z &= 10000 \text{ slug ft}^2 \end{aligned} \quad (140)$$

These values can be achieved by a suitable distribution of a total vehicle mass of 750 lbs. This choice of moments of inertia satisfies the physical restrictions as given by Eqs. (186)-(188) in the Appendix.

From Eqs. (20), (21) and (22), the values of R_x , R_y and R_z are obtained as

$$R_x = .9474$$

$$R_y = .5$$

$$R_z = .85$$

It is seen that these values also satisfy the conditions for pitch, yaw and roll stability as specified by Eqs. (192), (195) and (196).

By means of Eqs. (191), the characteristic frequency in pitch is found to be

$$\omega_{\alpha} = 1.5969 \omega_0 \quad (142)$$

while Eqs. (193) and (194) give the characteristic yaw-roll frequencies as

$$\omega_1 = 1.9531 \omega_0 \quad (143)$$

$$\omega_2 = .7043 \omega_0 \quad (144)$$

EFFECT OF AN IMPULSE MOMENT

The expressions for the attitude perturbations due to an impulse moment were determined in Section II and will be applied here to the case of the impulse resulting from meteoroid impact.

Pitch Impulse

The maximum pitch attitude perturbation due to an impulse moment is given by Eq. (30), which when combined with Eq. (16) gives

$$\alpha_m = .13225 \frac{r_c^{3/2} I_p}{R_z^{1/2} I_z} \quad (145)$$

where the units are as follows

$$r_c - n \text{ mi}$$

$$I_p - \text{lb-ft-sec}$$

$$I_z - \text{slug ft}^2$$

$$\alpha_m - \text{deg}$$

If this result is applied to the case of meteoroid impact, the impulse can be expressed in the form

$$I_p = r_p \Delta(mV) \quad (146)$$

where $\Delta(mV)$ is the change of momentum of the meteoroid, which is equal to the force impulse in lb-sec, while r_p is the pitch moment arm of this force in ft.

The determination of the momentum change involved in meteoroid impact is extremely complicated and involves not only the mass and velocity of the incident particle but also the mass and velocity of the vaporized target material. For the purposes of this Memorandum, the momentum change is evaluated as the product of the mass and velocity of the incident particle. This assumes an inelastic impact and gives a lower limit for the value of $\Delta(mV)$.

As an example, suppose the satellite vehicle is at synchronous altitude ($r_c = 22750$ n mi) and is subjected to a meteoroid impact for which

$$\Delta(mV) = 10^{-3} \text{ lb-sec}$$

$$r_p = 10 \text{ ft}$$

The resulting pitch attitude perturbation has a maximum value given by Eqs. (145) and (146) as

$$\alpha_m = .492 \text{ deg}$$

It should be noted that a meteoroid impact corresponding to as much as 10^{-3} lb-sec would occur relatively infrequently.

Roll Impulse

The maximum roll attitude perturbation due to an impulse moment in roll is given by Eq. (41) which can be expressed as

$$\beta_m = .22901 \frac{r_c^{3/2} I_R}{I_x} F_1(\omega_1, \omega_2) \quad (147)$$

where

$$F_1(\omega_1, \omega_2) = \frac{R_y + \frac{\omega_1 \omega_2}{2}}{\frac{\omega_1 \omega_2}{2} \left(\frac{\omega_1}{\omega_0} + \frac{\omega_2}{\omega_0} \right)} \quad (148)$$

and the units are the same as those used in the pitch case above.

As indicated in Section III, the coupling between yaw and roll results in a yaw response even though I_R is applied about the roll axis. The amplitude of this yaw response is given by Eq. (42), which can be written in the form

$$\psi_m = .4580 \frac{r_c^{3/2} I_R}{I_x} F_2(\omega_1, \omega_2) \quad (149)$$

where

$$F_2(\omega_1, \omega_2) = - \frac{(R_y - 1)}{\left(\frac{\omega_1}{\omega_0}^2 - \frac{\omega_2}{\omega_0}^2 \right)} \quad (150)$$

and

$$I_R = r_R \Delta(mV) \quad (151)$$

As an example, suppose that a meteoroid impact occurs on the same synchronous satellite and produces an impulse moment in roll. If I_R is specified by

$$\Delta(mV) = 10^{-3} \text{ lb-sec}$$

$$r_R = 10 \text{ ft}$$

where r_R is the roll moment arm, then the resulting amplitudes of the roll and yaw perturbations as determined by Eqs. (147) through (150) are given by

$$\beta_m = .425 \text{ deg}$$

$$\psi_m = .249 \text{ deg}$$

Yaw Impulse

The maximum yaw attitude perturbation due to an impulse moment in yaw is given by Eq. (48), which can be expressed as

$$\psi_m = .22901 \frac{r_c^{3/2} I_y}{I_x} F_3(\omega_1, \omega_2) \quad (152)$$

where

$$F_3(\omega_1, \omega_2) = \frac{4R_x + \frac{\omega_1 \omega_2}{2}}{\frac{\omega_1 \omega_2}{2} \left(\frac{\omega_1}{\omega_o} + \frac{\omega_2}{\omega_o} \right)} \quad (153)$$

The amplitude of the associated roll response is given by Eq. (49), which can be expressed as

$$\beta_m = .4580 \frac{r_c^{3/2} I_y}{I_x} F_4(\omega_1, \omega_2) \quad (154)$$

where

$$F_4(\omega_1, \omega_2) = \frac{R_x - 1}{\left(\frac{\omega_1}{2} - \frac{\omega_2}{2} \right) \left(\frac{\omega_1}{\omega_o} - \frac{\omega_2}{\omega_o} \right)} \quad (155)$$

and

$$I_Y = r_Y \Delta(mV) \quad (156)$$

As an example, if an impulse moment due to a meteoroid impact is characterized by

$$\Delta(mV) = 10^{-3} \text{ lb-sec}$$

$$r_Y = 10 \text{ ft}$$

the resulting yaw and roll amplitudes as determined by Eqs. (152) through (156) are

$$\psi_m = 11.11 \text{ deg}$$

$$\beta_m = .249 \text{ deg}$$

A comparison of the results for roll and yaw impulses shows that the largest response is that in yaw due to a yaw impulse. It is also seen that β_m due to a yaw impulse is equal to ψ_m due to a roll impulse. This can be demonstrated by a comparison of Eqs. (42) and (49).

EFFECT OF A STEP MOMENT

The expressions for the attitude perturbations due to a step increase in the applied moment were determined in Section II, and will be applied here to the case of the step change in the radiation pressure moment which occurs as the vehicle emerges from the earth's shadow.

Step Moment in Pitch

The steady-state deflection in pitch due to a step moment in pitch is given by Eq. (54), which can be rewritten as

$$\alpha_{ss} = 3.051 \times 10^{-4} \frac{r_c^3 M_p}{R_z I_z} \quad (157)$$

where the units are as follows

$$r_c = n \text{ mi}$$

$$I_z = \text{slug ft}^2$$

$$M_p = \text{lb-ft}$$

$$\gamma_{ss} = \text{deg}$$

To determine the magnitude of M_p , assume the radiation is incident along the x axis, and that the satellite presents an area of $A \text{ ft}^2$ in the x direction. If the surface is totally reflecting, the total force due to radiation pressure ($p_o = 1.865 \times 10^{-7} \text{ lbs/ft}^2$) is

$$F_R = p_o A \quad (158)$$

acting along the x axis. If the center of pressure is displaced a distance r_p up the y axis, the resulting pitch moment is expressed as

$$M_p = p_o A r_p \quad (159)$$

As an example, if the subtended area is 100 ft^2 and the center of pressure-center of mass offset, r_p , is $.1 \text{ ft}$, then the pitch moment is given by

$$M_p = 1.865 \times 10^{-6} \text{ lb-ft}$$

If a step change in moment of this magnitude were applied to the assumed satellite configuration in a synchronous orbit, the resulting steady-state pitch angle based on Eq. (157) would be

$$\alpha_{ss} = .788 \text{ deg}$$

Step Moment in Roll

The steady-state deflection in roll due to a step moment in roll is given by Eq. (61), which can be written in the form

$$\theta_{ss} = 2.288 \times 10^{-4} \frac{r_c^3 M_R}{I_z - I_y} \quad (160)$$

In addition, the amplitude of the associated yaw response is given by Eq. (60), which can be expressed as

$$\psi_m = 9.153 \times 10^{-4} \frac{r_c^3 M_R}{I_x} G_1(\omega_1, \omega_2) \quad (161)$$

where

$$G_1(\omega_1, \omega_2) = \frac{\frac{R_y}{R_x} - 1}{\frac{\omega_1 \omega_2}{\omega_c^2} \left(\frac{\omega_1}{\omega_0} - \frac{\omega_2}{\omega_0} \right)} \quad (162)$$

If the step moment M_R has a magnitude equal to that used for M_p , then the resulting response of the satellite as described by Eqs. (160)-(162) is

$$\theta_{ss} = .558 \text{ deg}$$

$$\psi_m = .616 \text{ deg}$$

Step Moment in Yaw

The steady-state deflection in yaw due to a step moment in yaw is given by Eq. (68), which can be rewritten in the form

$$\psi_{ss} = 9.153 \times 10^{-4} \frac{r_c^3 M_Y}{I_z - I_x} \quad (163)$$

The amplitude of the associated roll oscillation is given by Eq. (67), which can be expressed as

$$\beta_m = 9.153 \times 10^{-4} \frac{r_c^3 M_Y}{I_y} G_2(\omega_1, \omega_2) \quad (164)$$

where

$$G_2(\omega_1, \omega_2) = \frac{\frac{R_x}{w_0^2} - 1}{\frac{\omega_1 \omega_2}{w_0^2} \left(\frac{\omega_1}{w_0} - \frac{\omega_2}{w_0} \right)}$$

If the amplitude of the step moment in yaw is the same as that assumed for the other two axes, Eqs. (163) and (164) give the following results

$$\psi_{ss} = 40.2 \text{ deg}$$

$$\beta_m = .616 \text{ deg}$$

Again, it is seen that the system is extremely sensitive in its yaw response and exceeds the small angle assumption used in the derivation of the equations of motion. In addition, the yaw amplitude resulting from a step moment in roll is equal to the roll amplitude resulting from a step moment in yaw. This can be shown by a comparison of Eqs. (60) and (67).

In general, these results emphasize the importance of maintaining the center of radiation pressure of the vehicle close to the center of mass.

EFFECT OF STATION-KEEPING PROPULSION

In many satellite applications it may be necessary to make small adjustments in velocity in order to maintain station relative to other

satellites or, as in the case of a synchronous satellite, to keep the same position relative to the earth. If the thrust used to produce these velocity changes has a line of action which passes through the vehicle center of mass, no attitude disturbance will result. However, determination of the permissible misalignment is an important design problem.

Impulsive Velocity Change

It is assumed that station-keeping adjustments are restricted to velocity changes along the orbit. Thus, the resulting moments are about the pitch or yaw axes. If the velocity adjustment is made impulsively, then the amplitude of the resulting pitch oscillation can be determined by Eqs. (145) and (146). However, in this case, the quantity $\Delta(mV)$ represents the desired momentum change of the vehicle and r_p is a measure of the thrust misalignment. Similarly, the amplitude of the yaw response and its associated roll response are given by Eqs. (152)-(156) with the above definition of $\Delta(mV)$.

As an example, suppose that the assumed satellite vehicle, weighing 750 lbs, is in a synchronous orbit with a radius of 22,750 n mi. If a velocity change of 1 ft/sec is required and the moment arm, r_p , of the thrust about the pitch axis is equal to 10^{-3} ft, then the amplitude of the resulting pitch oscillation as given by Eq. (145) is

$$\alpha_m = 1.146 \text{ deg}$$

Similarly, if the thrust has a moment arm, r_y , about the yaw axis equal to 10^{-3} ft, then the amplitudes of the resulting yaw and roll oscillations as given by Eqs. (152) and (154) are

$$\psi_m = 25.57 \text{ deg}$$

$$\theta_m = .58 \text{ deg}$$

The assumed moment arm of 10^{-3} ft represents a precision of thrust alignment which is not achievable in practice.

Constant Thrust Application

In Ref. 4 the station-keeping propulsion requirements resulting from the ellipticity of the earth's equator was determined as 17 ft/sec/yr. However, based on observations of Syncom 2, a more realistic value appears to be 6 ft/sec/yr. For the assumed 750-lb satellite this requirement is equivalent to a steady thrust of 5.17×10^{-6} lb. If the moment arm, r_p , of this force about the pitch axis is .1 ft, then the resulting steady-state pitch angle can be determined by means of Eq. (157) as

$$\alpha_{ss} = .218 \text{ deg}$$

Similarly, if the moment arm, r_y , of this force about the yaw axis is .1 ft, then the resulting steady-state yaw angle as given by Eq. (163) is

$$\psi_{ss} = 11.14 \text{ deg}$$

Thus, it is seen that even at extremely low thrust levels, appreciable attitude disturbances can occur.

EFFECT OF ORBITAL ECCENTRICITY

The amplitude of the forced oscillation in pitch due to orbital eccentricity is given by Eq. (78), which can be expressed as

$$\alpha_m = 114.6 \frac{\epsilon}{3R_z - 1} \quad (166)$$

where α_m is in degrees.

As an example, if the assumed satellite configuration is on an orbit with an eccentricity of .005, Eq. (166) gives a pitch amplitude of

$$\alpha_m = .370 \text{ deg}$$

An examination of Eq. (78) shows that the value of α_m depends only on the satellite configuration and the orbital eccentricity, and is independent of the orbital period. It is seen that α_m in radians is approximately equal to the eccentricity; thus a low eccentricity orbit is essential in order to avoid large errors in pitch attitude.

EFFECT OF ROTATING EQUIPMENT

In Section II, the expressions were developed for the attitude perturbations which would result due to the presence of rotating equipment on-board a satellite. These relations are used here to determine representative values of these perturbations for the assumed satellite configuration.

Rotation about Pitch Axis

From Eq. (104) it is seen that if a rotor of moment of inertia I_s is accelerated to a steady rotation rate of $\dot{\alpha}_{2ss}$ relative to the main vehicle, an oscillation in pitch is initiated which has an amplitude given by

$$\begin{aligned} \alpha_m &= \frac{\dot{\alpha}_{2ss} I_s}{\omega_\alpha I_z} \\ &\approx \frac{G_p}{\omega_\alpha I_z} \end{aligned} \quad (167)$$

where G_p is the angular momentum of the rotor. Equation (167) can be rewritten as

$$\alpha_m = .13221 \frac{r_c^{3/2} G_p}{R_z^{1/2} I_z} \quad (168)$$

where the units are

$$r_c = \text{in mi}$$

$$I_z = \text{slug ft}^2$$

$$G_p = \text{slug ft}^2/\text{sec}$$

$$\alpha_m = \text{deg}$$

The magnitude of G_p is given by the relation

$$G_p = .05236 I_s \dot{\alpha}_{2ss} \quad (169)$$

where the units are

$$I_s = \text{slug ft}^2$$

$$\dot{\alpha}_{2ss} = \text{rpm}$$

$$G_p = \text{slug ft}^2/\text{sec}$$

As an example, suppose the assumed satellite configuration is on a synchronous orbit and has a rotor with $I_s = .001 \text{ slug ft}^2$ and a rotation rate $\dot{\alpha}_2 = 100 \text{ rpm}$

By means of Eqs. (168) and (169) the amplitude of the resulting pitch oscillation is found to be

$$\alpha_m = .258 \text{ deg}$$

It should be noted that if an adequate damping system is present, this pitch oscillation will be reduced to zero. However, each time the rotor starts or stops a similar transient oscillation will be generated.

Rotation about Yaw Axis

If the internal rotation is about the yaw axis, the resulting yaw response is oscillatory with an amplitude determined by Eq. (126), which can be written in the form

$$\dot{\psi}_m = .22904 \frac{r_c^{3/2} G_Y}{I'_y} H_1(\omega_1, \omega_2) \quad (170)$$

where

$$H_1(\omega_1, \omega_2) = \frac{1 + 3R_x + \frac{\omega_1 \omega_2}{2}}{\frac{\omega_1 \omega_2}{\omega_0^2} \left(\frac{\omega_1}{\omega_0} + \frac{\omega_2}{\omega_0} \right)} \quad (171)$$

and

$$G_Y = .05236 I_s \dot{\psi}_{2ss} \quad (172)$$

In addition, the associated roll response consists of a steady-state deflection plus an oscillatory term with an amplitude equal to the steady-state term. The magnitude of the steady-state term is given by Eq. (124), which can be written as

$$\dot{\psi}_{ss} = .05726 \frac{r_c^{3/2} G_Y}{I'_z - I'_y} \quad (173)$$

If the value of G_Y is identical to that chosen for G_P , the resulting yaw and roll responses of the assumed synchronous satellite are given by

$$\psi_m = 5.876 \text{ deg}$$

$$\beta_m = .114 \text{ deg}$$

$$\beta_{ss} = .114 \text{ deg}$$

As in the pitch case, suitable damping will reduce the oscillatory amplitudes to zero. However, the steady-state roll angle is unaffected as long as the internal rotation continues.

Rotation about Roll Axis

If the internal rotation is about the roll axis, the resulting roll response is oscillatory with an amplitude given by Eq. (139), which can be written in the form

$$\beta_m = .22904 \frac{r_c^{3/2} G_R}{I_x'} H_2(\omega_1, \omega_2) \quad (174)$$

where

$$H_2(\omega_1, \omega_2) = \frac{1 + \frac{\omega_1 \omega_2}{2}}{\frac{\omega_1 \omega_2}{2} \left(\frac{\omega_1}{\omega_0} + \frac{\omega_2}{\omega_0} \right)} \quad (175)$$

and

$$G_R = .05236 I_s \dot{\beta}_{2ss} \quad (176)$$

In addition, the associated yaw response is made up of a steady-state term given by Eq. (137), which can be expressed as

$$\psi_{ss} = .22904 \frac{r_c^{3/2} G_R}{I'_z - I'_x} \quad (177)$$

and also an oscillatory term with an amplitude given by Eq. (138), which can be put in the form

$$\psi_m = .22904 \frac{r_c^{3/2} G_R}{I'_z - I'_x} H_3(\omega_1, \omega_2) \quad (178)$$

where

$$H_3(\omega_1, \omega_2) = \frac{2 \left(\frac{\omega_1^2}{\omega_0^2} + \frac{\omega_2^2}{\omega_0^2} \right) - \frac{\omega_1^2 \omega_2^2}{\omega_0^4}}{2 \left(\frac{\omega_1^2}{\omega_0^2} - \frac{\omega_2^2}{\omega_0^2} \right)} \quad (179)$$

If the value of G_R is the same as that assumed for G_p and G_y , the yaw-roll response of the satellite as given by Eqs. (174), (177) and (178) is as follows

$$\beta_m = .282 \text{ deg}$$

$$\psi_m = 8.34 \text{ deg}$$

$$\psi_{ss} = 8.23 \text{ deg}$$

As in the previous cases, the presence of damping will reduce the oscillatory amplitudes to zero after a few orbits, but the steady-state yaw angle will remain as long as the rotor maintains its steady-state rate $\dot{\beta}_{2ss}$.

IV. DISCUSSION AND CONCLUSIONS

An examination of the results presented in Section III indicates that a satellite which depends on gravity gradient to maintain attitude stability may be subject to appreciable perturbations. This results from the fact that although the principal moments of inertia can be chosen so that the vehicle is inherently stable, the restoring torques are extremely small. Thus, it is necessary to consider the effects of very small disturbances such as meteoroid impact, radiation pressure, station-keeping propulsion, orbital eccentricity and internal rotating equipment which would ordinarily be ignored in the presence of an active attitude control system.

The results indicate that a meteoroid impact capable of producing an impulsive moment equal to .01 lb-ft-sec can produce deflections of a synchronous satellite of the order of a fraction of a degree in either pitch or roll and of several degrees in yaw. The frequency with which such an impact might occur depends on the mechanism assumed for hyper-velocity impact and a more precise description of the vehicle configuration. Thus, the results are given in terms of the momentum change occurring at impact.

In the case of radiation pressure it is seen that for the assumed satellite configuration at synchronous altitude, a moment of 1.865×10^{-6} lb-ft due to radiation pressure could cause deflection of the order of a fraction of a degree in pitch and roll and tens of degrees in yaw. The assumed moment corresponds to the force on 100 ft^2 of reflecting surface with a moment arm of .1 ft. This emphasizes the importance of maintaining symmetry of the reflecting properties of the vehicle.

Impulsive station-keeping propulsion requires extreme precision in the alignment of the thrust axis through the center of mass. Even for extremely low thrust levels and long burning times the precision is still excessive. Thus, it appears inadvisable to use a gravity gradient system if station-keeping is required.

Orbital eccentricity causes an oscillation in pitch of .37 deg for an eccentricity of .005. This eccentricity corresponds to a difference of about 225 n mi in apogee and perigee altitudes for a synchronous orbit. Actually, Syncom and Early Bird have achieved appreciably lower eccentricities.

In the case of rotating equipment, it is seen that an internal piece of rotating equipment with an angular momentum of .005236 slug- ft^2/sec aligned along the pitch axis produces an oscillatory deflection in pitch of about .25 deg. The same equipment aligned along the yaw axis produces a steady-state roll angle of about .11 deg and a yaw oscillation of 5.88 deg. Finally, if the axis of rotation is along the roll axis a steady-state yaw angle of 8.23 deg is produced as well as a roll oscillation of about .28 deg.

To summarize the results presented above, the various sources of perturbation are listed below in the order of their importance and with indications of how their effects can be reduced.

1. Station-Keeping Propulsion

It is extremely doubtful that the required precision of thrust alignment can be achieved in practice. Thus, station-keeping is incompatible with gravity gradient stabilization, unless auxiliary sensors are used to determine the attitude.

2. Rotating Equipment

If such equipment is essential to the success of the mission, its perturbing effect can be reduced by utilizing similar counter-rotating elements. Thus, the quantities G_p , G_y , and G_r are the residual uncompensated angular momenta.

3. Radiation Pressure

In order to minimize the effects of radiation pressure, it is necessary to maintain geometrical symmetry as well as symmetry of the surface reflectivity so that the force due to radiation acts through the center of mass of the vehicle. This may be difficult to achieve if the vehicle is subject to thermal bending and resultant variations in the position of the center of mass.

4. Meteoroid Impact

The relative importance of the effect depends upon the statistical distribution of meteoroid impacts as a function of impulse as well as a detailed description of the vehicle configuration, both of which are beyond the scope of this Memorandum.

5. Orbital Eccentricity

This effect requires that the vehicle be on a nearly circular orbit to avoid a steady oscillation in pitch. However, eccentricities currently being achieved are low enough that this effect should not be serious.

Finally, it should be pointed out that for many applications attitude variations of the vehicle of the order of a few degrees may be acceptable if these variations can be measured precisely and read out to the user.

Appendix

DETERMINATION OF VEHICLE CONFIGURATION AND STABILITY

In Section II of this Memorandum, the linearized equations of motion were developed in terms of the parameters R_x and R_y (see Eqs. (23)-(26)), and it was pointed out that there are certain restrictions on the permissible values of R_x and R_y .

PHYSICAL RESTRICTION

The fact that the three principal moments of inertia cannot be selected arbitrarily puts a limitation on the values of R_x and R_y through Eqs. (20) and (21) as follows.

The three moments of inertia can be expressed in the form

$$I_x = \int_M y^2 dm + \int_M z^2 dm \quad (180)$$

$$I_y = \int_M x^2 dm + \int_M z^2 dm \quad (181)$$

$$I_z = \int_M x^2 dm + \int_M y^2 dm \quad (182)$$

If these three equations are solved simultaneously for the three integrals, their values are given by

$$2 \int_M x^2 dm = -I_x + I_y + I_z \quad (183)$$

$$2 \int_M y^2 dm = I_x - I_y + I_z \quad (184)$$

$$2 \int_M z^2 dm = I_x + I_y - I_z \quad (185)$$

Since the left sides of these equations are always positive, the following conditions must be satisfied by the moments of inertia

$$I_y + I_z \geq I_x \quad (186)$$

$$I_z + I_x \geq I_y \quad (187)$$

$$I_x + I_y \geq I_z \quad (188)$$

Thus, it is seen that the sum of any two principal moments of inertia is greater than or equal to the third.

By means of the above inequalities and the definitions of R_x and R_y , Eqs. (20) and (21), it can be shown that

$$-1 \leq R_x \leq +1 \quad (189)$$

$$-1 \leq R_y \leq +1 \quad (190)$$

Thus, in the R_x , R_y plane, all physically realizable configurations lie inside of the square shown in Fig. 2.

STABILITY RESTRICTION

In addition to the physical restriction specified above, it is also necessary to rule out combinations of R_x and R_y for which the vehicle is unstable in either pitch, yaw or roll.

Pitch Stability

An examination of Eq. (25) shows that the pitch response is completely uncoupled from the yaw and roll motion and has a characteristic

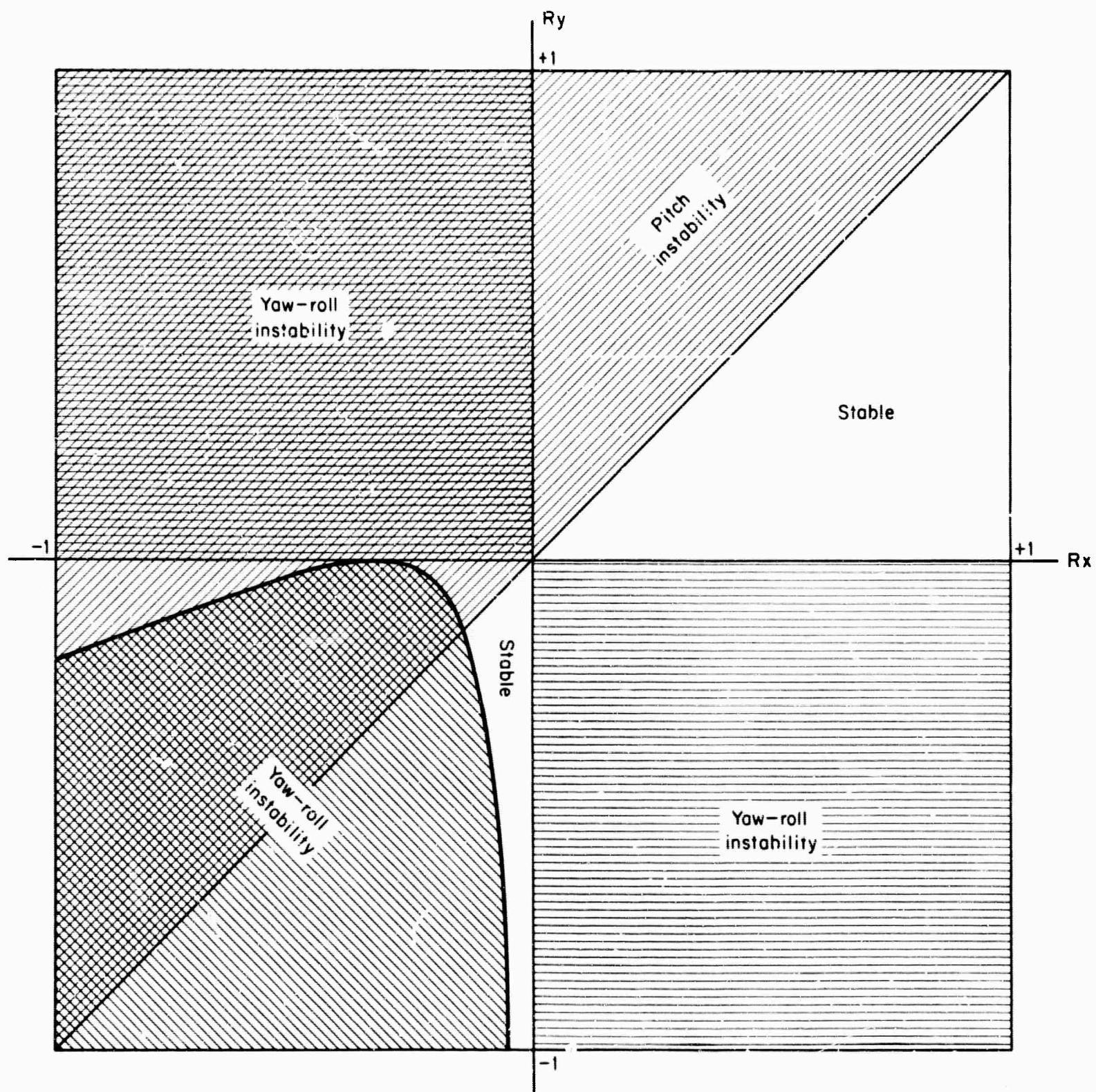


Fig.2—Stability region in the R_x , R_y plane

frequency given by

$$\omega_{\alpha}^2 = 3R_z \omega_0^2 = \frac{3(R_x - R_y) \omega_0^2}{1 - R_x R_y} \quad (191)$$

Since the absolute values of R_x and R_y are less than unity, Eq. (191) represents an additional restriction such that

$$R_x \geq R_y \quad (192)$$

in order that the vehicle be stable in pitch. This condition rules out the region in the R_x , R_y plane above the 45° line ($R_x = R_y$) (see Fig. 2).

Yaw-Roll Stability

An examination of Eqs. (23) and (24) shows that the yaw and roll responses are coupled and if the external moments M_x and M_y are set equal to zero, the two natural frequencies for the yaw-roll motion can be determined from the characteristic equation as

$$\frac{\omega_1^2}{\omega_0^2} = \frac{(3R_x + R_x R_y + 1) + \sqrt{(3R_x + R_x R_y + 1)^2 - 16 R_x R_y}}{2} \quad (193)$$

$$\frac{\omega_2^2}{\omega_0^2} = \frac{(3R_x + R_x R_y + 1) - \sqrt{(3R_x + R_x R_y + 1)^2 - 16 R_x R_y}}{2} \quad (194)$$

These two equations impose additional restrictions on R_x and R_y since it is necessary that ω_1^2 and ω_2^2 be positive and real. The first of these conditions is satisfied if R_x and R_y have the same sign, that is

$$R_x R_y \geq 0 \quad (195)$$

This condition rules out the second and fourth quadrant of the R_x , R_y plane although the second quadrant has already been ruled out by Eq. (192).

The requirement that ω_1^2 and ω_2^2 be real is satisfied if

$$(3R_y + R_x R_y + 1)^2 \geq 16 R_x R_y \quad (196)$$

which rules out the region in the third quadrant of the R_x , R_y plane below the curve

$$(3R_x + R_x R_y + 1)^2 = 16 R_x R_y \quad (197)$$

as shown in Fig. 2.

R_x , R_y PLANE

The physical and stability restrictions developed above have defined the permissible region of the R_x , R_y plane as shown in Fig. 2. This is similar to the result obtained by DeBra and Delp in Ref. 5.

The triangular portion of the region in the first quadrant corresponds to those configurations ordinarily considered for gravity gradient stability in which

$$I_z > I_x > I_y \quad (198)$$

The origin ($R_x = R_y = 0$) corresponds to a spherical mass distribution while the hypotenuse ($R_x = R_y$) represents configurations which have rotational symmetry about the z or pitch axis ($I_x = I_y$). The horizontal edge of the triangle ($R_y = 0$) corresponds to configurations

with rotational symmetry about the y or yaw axis ($I_z = I_x$). Finally, the vertical edge of the triangle ($R_x = 1$) represents any linear mass distribution along the y or yaw axis ($I_y = 0$).

It should be noted that the point $R_x = R_y = 1$ represents a planar mass distribution in the x, y plane ($I_x + I_y = I_z$). The exact nature of this mass distribution may be anything between a circular disc and a linear distribution along the y axis.

The other portion of the R_x , R_y plane which is permissible lies in the third quadrant as shown in Fig. 2. It is bounded by the negative R_y axis; the line, $R_x = R_y$; the line, $R_y = -1$; and the curve specified by Eq. (197). In this area the three moments of inertia have the following relation

$$I_x > I_y > I_z \quad (199)$$

with $I_x = I_y$ along the 45° line ($R_x = R_y$) and $I_y = I_z$ along the negative R_y axis ($R_x = 0$). It is seen that the stability in this case is that of an elongated body spinning about its axis of minimum moment of inertia at orbital angular rate.

Pitch Frequency Contours

It is of interest to determine the contours of constant pitch frequency on the R_x , R_y plane. This can be done by means of Eq. (191). The resulting contours are plotted in Fig. 3. It is seen that in both parts of the stable region the value of ω/ω_0 is limited to the range

$$0 \leq \frac{\omega}{\omega_0} \leq \sqrt{3} \quad (200)$$

The dotted portions of contours correspond to regions of yaw-roll instability. Thus, although the pitch motion is apparently stable, the

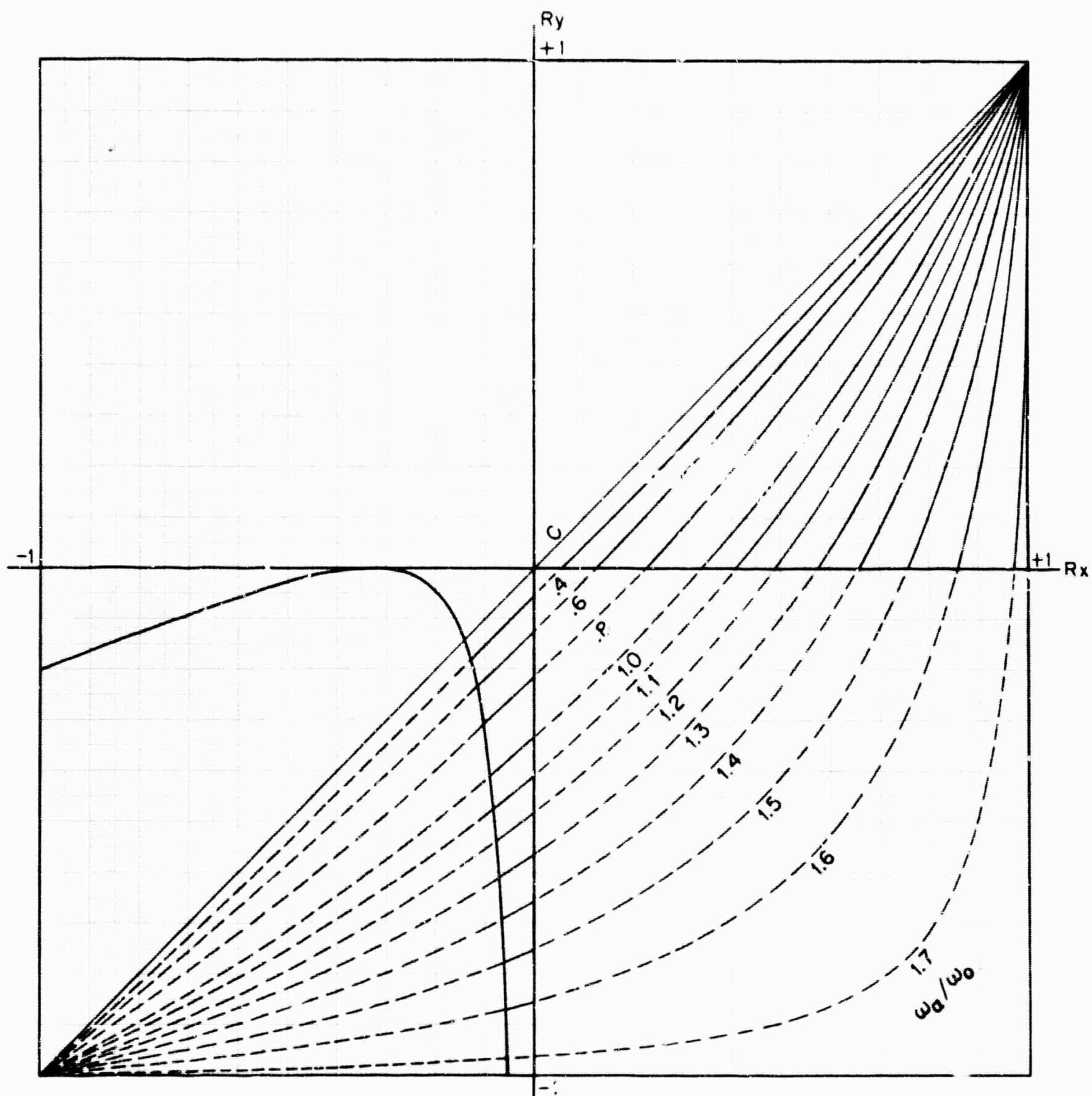


Fig. 3—Contours of constant pitch frequency

linearized equation describing the pitch motion, Eq. (25), would no longer be valid in the presence of large yaw and roll angles.

Yaw-Roll Frequency Contours

The characteristic yaw-roll frequencies specified by Eqs. (193) and (194) are roots of the equation

$$\frac{\frac{4}{\omega_0^4}}{\frac{2}{\omega_0^2}} - \frac{\frac{\omega}{2}}{\frac{\omega_0^2}{\omega_0}} (3R_x + R_x R_y + 1) + 4R_x R_y = 0 \quad (201)$$

This function can be used to determine contours of constant yaw-roll frequency in the R_x , R_y plane. These contours are plotted in Fig. 4 while Fig. 5 is an enlargement of the stable region in the third quadrant.

An examination of these two figures shows that through any point of the stable region of the R_x , R_y plane there are contours corresponding to two values of frequency. These two values are the ω_1 and ω_2 defined by Eqs. (193) and (194). It is seen that in the first quadrant

$$1 < \frac{\omega_1}{\omega_0} < 2 \quad (202)$$

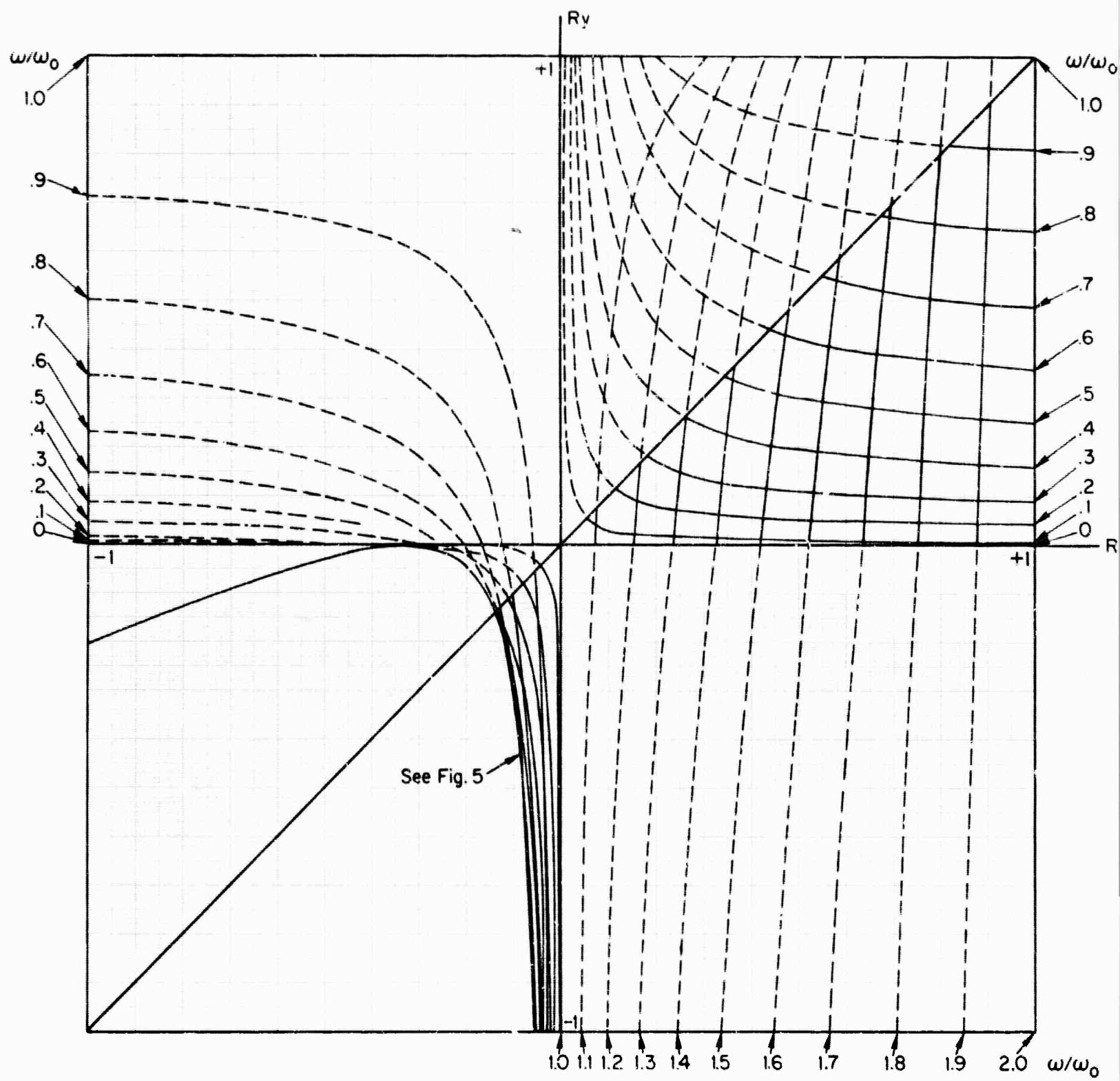
and

$$0 < \frac{\omega_2}{\omega_0} < 1 \quad (203)$$

while in the third quadrant both frequencies are less than ω_0 so that

$$\frac{\omega_2}{\omega_0} \leq \frac{\omega_1}{\omega_0} < 1 \quad (204)$$

It can also be shown that the limiting curve defined by Eq. (197) is the envelope of the frequency contours in this region and represents



the locus for which the two characteristic frequencies ω_1 and ω_2 are equal. The frequency values along this curve are indicated in Fig. 5.

DISCUSSION

The mapping of the R_x , R_y plane presented in Figs. 2 through 5 enables the user to determine whether or not a given configuration is stable and if stable, to determine the characteristic pitch, yaw and roll frequencies.

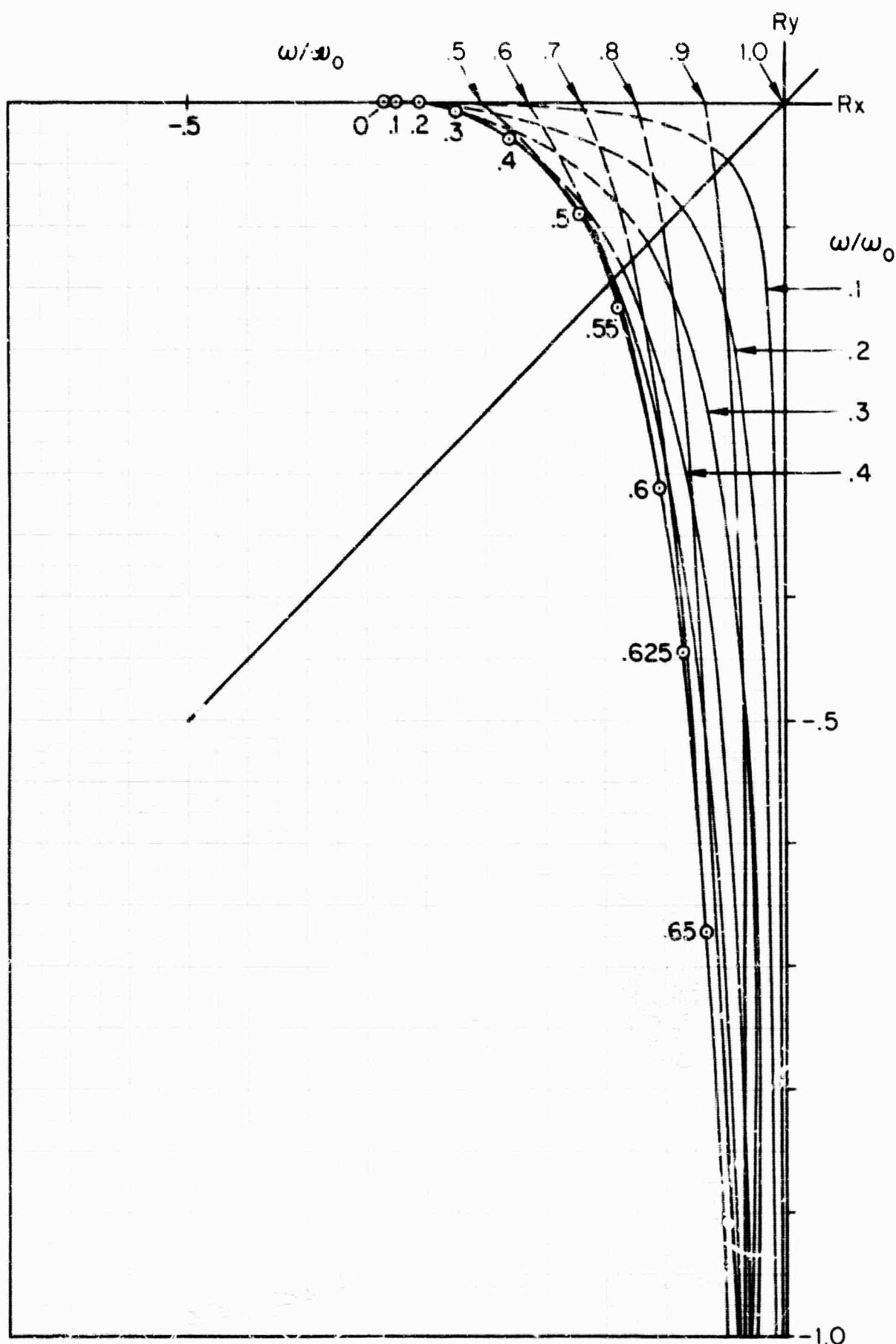


Fig. 5—Contours of constant yaw-roll frequency
(third quadrant)

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